Multiobjective Optimization in Structural Design with Uncertain Parameters and Stochastic Processes

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The application of multiobjective optimization techniques to structural design problems involving uncertain parameters and random processes is studied. The design of a cantilever beam with a tip mass subjected to a stochastic base excitation is considered for illustration. Several of the problem parameters are assumed to be random variables and the structural mass, fatigue damage, and negative of natural frequency of vibration are considered for minimization. The solution of this three-criteria design problem is found by using global criterion, utility function, game theory, goal programming, goal attainment, bounded objective function, and lexicographic methods. It is observed that the game theory approach is superior in finding a better optimum solution, assuming the proper balance of the various objective functions. The procedures used in the present investigation are expected to be useful in the design of general dynamic systems involving uncertain parameters, stochastic process, and multiple objectives.

Nomenclature

a = acceleration of the end mass, m/s^2
A = permissible acceleration of the end mass, m/s^2
A_j, B_j = maximum permissible values of α_j, β_j
b = inner width of the beam, m
c_1, c_2, c_3 = const
d = inner depth of the beam, m
D = expected fatigue damage of the beam
E = Young's modulus, Pa
f = vector of objective functions
f_j = jth objective function (f_1 in m^2, f_2 in rad/s, f_3 dimensionless)
F_i = ith standardized objective function
g_j = jth inequality constraint
g = vector of inequality constraints
h = wall thickness of the beam, m
h_j = upper bound on γ_j
I = moment of inertia of cross section, m^4
k = number of objective functions
$k_1, k_2, \ldots$ = const
K = set of objective function numbers
ℓ = length of the beam, m
ℓ_1 = number of equality constraints
ℓ_j = jth equality constraint
L = set of equality constraint numbers
m = end mass, kg
m_1 = number of inequality constraints
M = set of inequality constraint numbers
n = number of design variables
N_1 = number of cycles of stress
$p_j, p_k, p_1, p_2, \ldots$ = specified (permissible) values of probabilities
$P[\,]$ = probability of occurrence of event [ ]
R = vector of random variables
$\bar{R}$ = vector of mean values of random variables
s = maximum stress induced in the beam, Pa
s_f = flange buckling stress, Pa
s_w = web buckling stress, Pa
S = supercriterion, Eq. (40)
S_1 = amplitude of stress, Pa
S_y = yield stress, Pa
t = time parameter, s
T = specified time duration, s
X = set of feasible design points
X = vector of design variables $= \{x_1 x_2 x_3\}^T \equiv \{\bar{b}\bar{d}\bar{h}\}^T$
X_0 = starting design vector
$X^{(1)}, X^{(2)}$ = typical feasible design points
$Y(t)$ = displacement of the end mass, m
$\dot{Y}(t)$ = velocity of the end mass, m/s
$\ddot{Y}(t)$ = base acceleration, m/s^2
w_i = const
α_j, β_k = random response quantities
γ_j = deterministic response quantity
$\Gamma(\,)$ = gamma function
ζ = damping ratio, dimensionless
ν_a^+ = crossing rate for acceleration a
ρ = density of the material, kg/m^3
$\sigma_{(\,)}$ = standard deviation of ( )
Φ_0 = power spectral density of base excitation, m^2/s^3 · rad
ω = natural frequency of the beam, rad/s
$\{\,\}^T$ = transpose of { }
$(\bar{\ })$ = mean value of ( )

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Introduction

MODERN computer-aided design methods for engineering systems generally assume that a scalar objective function or functional such as cost, efficiency, or weight can be defined, so that standard computational algorithms from mathematical programming or optimal control can be applied. The usefulness of these methods is seriously limited by the fact that the quality of a complex engineering system generally depends on a number of different and often conflicting objective functions which cannot be combined into a single design objective. Hence, the consideration of multiple objectives becomes important in the optimization of engineering systems.

Although several methods have been proposed in the mathematical programming literature over the last decade for the solution of multiobjective optimization problems, the applicability of most of the methods was demonstrated through simple linear model problems and reported examples of applications to practical nonlinear problems seem to be rare. The presence of uncertain parameters and stochastic processes in practical dynamic systems make the multiobjective optimization problem even more complex.

This paper presents the application of multiobjective optimization techniques to the design of a simple structural problem involving uncertain parameters and stochastic processes. The necessity of optimizing the structural systems involving dynamic restrictions, random parameters, stochastic processes, and multiple objectives has been amply demonstrated.[1-3] The multiobjective design of a cantilever beam carrying a mass is considered for illustration. The geometric and material properties of the beam are treated as probabilistic quantities and the base of the beam is assumed to be subjected to a Gaussian white noise excitation. Along with several constraints, the minimization of the structural mass, the maximization of the natural frequency of vibration, and the minimization of the fatigue damage of the beam are considered as objectives.

Since the design parameters are random, the objective functions and the response parameters of the problem will also be probabilistic in nature. For this type of problems, three types of constraints can be identified,[4] as follows:

1) Dynamic probabilistic constraints involving both the random variables and the stochastic process that can be written as

$$P[\beta_j(X,R,t) > B_j] \leq p_j, \quad 0 \leq t \leq T \tag{1}$$

According to this inequality, the probability of a random dynamic response quantity β_j (such as the stress induced in the structure) exceeding a specified quantity B_j during the time interval $0 \leq t \leq T$ must be less than or equal to a permissible small value p_j.

2) Probabilistic constraints involving only the random variables that can be stated as

$$P[\alpha_j(X,R) > A_j] \leq p_j \tag{2}$$

This constraint states that the probability of a random response quantity α_j (such as the buckling stress) exceeding a specified upper limit A_j must be less than or equal to the permissible value p_j.

3) Deterministic constraints that can be expressed in the form

$$\gamma_j(X,\bar{R}) \leq h_j \tag{3}$$

This implies that the deterministic response quantity γ_j, which depends on the design vector X and the mean values of the random variables, must be less than or equal to a specified value h_j.

In this work, the constraints of Eq. (2) are converted into equivalent deterministic form by assuming a known probabilistic distribution for the random variables in conjunction with a linearization technique.[4] Similarly, in Eq. (1), first the time parameter is eliminated by assuming the excitation to be a stationary process (so that the mean and standard deviations of the response become independent of time) and then by idealizing the system as a single degree-of-freedom system for random vibration analysis. Although these simplifications are applied in the formulation of the specific beam problem, it is expected that similar procedures are applicable for any general dynamic system involving uncertain parameters and the stochastic process.

The resulting three-criteria beam design problem is solved by using some of the multiobjective programming techniques that were until now used primarily for the solution of linear problems. More specifically, the global criterion, utility function, game theory, goal programming, goal attainment, bounded objective function, and lexicographic methods are used.[5-7] The relative efficiencies of these methods are studied for the solution of multiobjective design problems involving uncertain parameters and stochastic process. The other available methods, such as insensitive design procedures[8] and heuristic techniques,[9-11] are not considered in this work.

General Multiobjective Optimization Problem

A multiobjective or vector minimization problem can be stated as follows:

$$\text{Minimize } f(X)$$

subject to

$$g_j(X) \leq 0 \quad \forall j \in M = \{1,2,\ldots,m_1\}$$
$$\ell_j(X) = 0 \quad \forall j \in L = \{1,2,\ldots,\ell_1\} \tag{4}$$

where

$$X = \{x_1, x_2, \ldots, x_n\}^T$$
$$f(X) = \{f_1(X), f_2(X), \ldots, f_k(X)\}^T$$

For a single criterion optimization problem, the optimum point is defined as one that minimizes the objective function $f(X)$ subject to the constraints $g_j(X) \leq 0$, $j \in M$, and $\ell_j(X) = 0$. Attempting to define a vector minimal point as one at which all components of the objective function vector are simultaneously minimized would not be an adequate generalization since such "utopia" points[12] are seldom attainable. For example, the point X that minimizes $f_1(X)$ need not minimize $f_2(X)$.

The transformation of the problem into a scalar problem by minimizing some scalar-valued function of the vector objective could be a powerful technique for finding the vector minima once a solution concept has been defined. The fundamental problem is to formulate a definition of a vector minimum when the components of the objective vector have different measures or units. There are a number of rationales that can be used to approach this problem. The most widely used rationale, studied by several recent authors, is due to the concept of an undominated or Pareto-minimal solution.[13-16] According to this concept, the objective vector $f(X^{(1)})$ evaluated at $X^{(1)} \in X$ dominates or is better than the objective vector $f(X^{(2)})$ evaluated at $X^{(2)} \in X$ if

$$f_k(X^{(1)}) \leq f_k(X^{(2)}) \; \forall k \in K = \{1,2,\ldots,k\}$$

and

$$f_j(X^{(1)}) < f_j(X^{(2)}) \tag{5}$$

for at least one $j \in K$. The game theory, goal programming, and goal attainment methods use the concept of a Pareto-optimal point, either directly or indirectly.

Formulation of the Example Problem

The optimum design of the cantilever beam, with a hollow rectangular cross section and tip mass as shown in Fig. 1, is considered for illustration. The material properties E, S_y, and ζ, the geometric parameters b, d, h, and ℓ, and the end mass m are assumed to be random variables. The expected values and coefficients of variation of E, S_y, ζ, ℓ, and m are assumed to be known. The base of the beam is assumed to be subjected to a Gaussian white noise excitation of constant-power spectral density Φ_0. This problem represents a simplfied model of a water tank subjected to an earthquake type of loading. The mean values of the cross-sectional dimensions b, d, and h are treated as the design variables x_1, x_2, and x_3, respectively. Three objective functions are considered, namely, the minimization of structural mass, the maximization of the

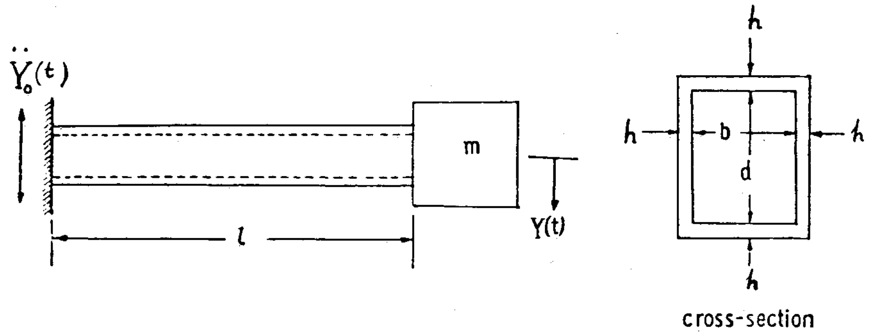

Fig. 1 Cantilever beam with tip mass.

natural frequency, and the minimization of the fatigue damage of the beam in a specified time T,

$$f_1(X) = \text{mean value of structural mass} \tag{6}$$

$$f_2(X) = -(\text{mean value of natural frequency}) \tag{7}$$

$$f_3(X) = \text{fatigue damage in time } T \tag{8}$$

Three types of constraints are considered in the problem formulation. Type (1) constraints are:

g_1: P [(maximum stress induced in the beam

$$\text{in } 0 \le t \le T) \ge S_y] \le p_1 \tag{9}$$

g_2: P [(maximum acceleration of the tip mass

$$\text{in } 0 \le t \le T) \ge A] \le p_2 \tag{10}$$

Type (2) constraints:

$$g_3: \ P\,[\text{flange buckling stress} \ge S_y] \le p_3 \tag{11}$$

$$g_4: \ P\,[\text{web buckling stress} \ge S_y] \le p_4 \tag{12}$$

$$g_5: \ P\,[d \ge (\ell/10)] \le p_5 \tag{13}$$

$$g_6: \ P\,[b \ge (\ell/10)] \le p_6 \tag{14}$$

g_7: standard deviation of mass is to be less than a fraction of its mean value, as

$$\sigma_{F_1} \le k_1 \bar{F}_1 \tag{15}$$

Type (3) constraints, the non-negativity of the design variables:

$$g_8: \ \bar{b} \ge 0 \tag{16}$$

$$g_9: \ \bar{d} \ge 0 \tag{17}$$

$$g_{10}: \ \bar{h} \ge 0 \tag{18}$$

Inequality (9) states that the probability of the maximum stress in the beam exceeding the yield stress in time T must be less than or equal to a specified small probability p_1. Inequality (10) states a similar constraint on the acceleration of the end mass. The requirement that the buckling stresses in the flange or web be less than or equal to the yield stress and the upper bounds on the dimensions b and d is stated in probabilistic form in Eqs. (11-14). Inequality (15) restricts the standard deviation of the first objective function (mass) to being less than a specified fraction of the mean value. Finally, the non-negativity restrictions on the design variables are stated in Eqs. (16-18). Although there are no equality constraints in the present formulation, their presence will not limit the applicability of the multicriteria optimization methods considered in this work.

The response quantities, in the deterministic form, can be expressed as:

$$\text{Structural mass of the beam} = 2\rho\ell h(b+d) \tag{19}$$

Since ρ and ℓ do not vary, the objective function f_1 is taken as

$$f_1 = 2\bar{h}(\bar{b}+\bar{d}) \tag{20}$$

$$\text{Natural frequency} = \omega = \left(\frac{\text{stiffness of beam}}{\text{mass at end}}\right)^{1/2} = \left(\frac{3EI}{m\ell^3}\right)^{1/2} \tag{21}$$

$$\text{Acceleration of end mass} = a = \ddot{Y}(t) \tag{22}$$

$$\text{Flange buckling stress} = s_f = k_2 E(h/b)^2 \tag{23}$$

$$\text{Web buckling stress} = s_w = k_3 E(h/d)^2 \tag{24}$$

where

$$I = (3bd^2h - 6bdh^2 + d^3h - 6d^2h^2)/6 \tag{25}$$

The standard deviations of the response quantities are given in the Appendix. The expected fatigue damage of the beam in time T can be obtained, assuming a narrow-band random process and the Palmgren-Miner hypothesis of fatigue failure, as[17]

$$D(X) = \frac{\nu_a^+ T}{\beta}(\sqrt{2}\sigma_s)^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \tag{26}$$

where the fatigue law has been assumed to be of the form

$$N_1 S_1^\alpha = \beta \tag{27}$$

where N_1 is the number of cycles, S_1 the magnitude of stress, and α and β constants. The crossing rate ν_a^+ is given by

$$\nu_a^+ = \frac{1}{2\pi} \cdot \frac{\sigma_{\dot{Y}}}{\sigma_Y} \cdot \exp\left(-\frac{a^2}{2\sigma_Y^2}\right) \tag{28}$$

with

$$\sigma_{\dot{Y}}^2 = \omega^2 \sigma_Y^2 \tag{29}$$

Numerical Results

For the numerical study of the multiobjective beam optimization problem, the data given in Table 1 are used. The starting design vector for all the multiobjective techniques is chosen as

Table 1 Problem data

End mass, $\bar{m}$, $\sigma_m = 175.2681, 17.52681$ kg
Length of beam, $\bar{\ell}$, $\sigma_\ell = 5.08, 0.0508$ m
Damping ratio, $\zeta = 0.02$
Yield stress of the material, $\bar{S}_y$, $\sigma_{S_y} = 137.8951, 13.78951$ MPa
Young's modulus, $\bar{E}$, $\sigma_E = 73.0844, 7.30844$ GPa
Permissible acceleration of end mass, $A = 10\,g = 98.1456$ m/s^2
Power spectral density of base excitation, $\Phi_0 = 0.15328$ m^2/s^3 · rad
Constants of fatigue law, $\alpha = 6.0$, $\beta = 6.4 \times 10^{31}$
Duration of excitation, $T = 3600$ s
Constants in Eqs. (15), (23), and (24), $k_1 = 0.2$, $k_2 = 21.72$, $k_3 = 3.62$
Permissible probabilities, $p_1 = p_2 = 0.001$; $p_j = 0.0124$, $j = 3,4,5,6$

$$X_0 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \bar{b} \\ \bar{d} \\ \bar{h} \end{Bmatrix} = \begin{Bmatrix} 0.25400 \\ 0.45720 \\ 0.00762 \end{Bmatrix} m$$

which corresponds to $f_1 = 0.010838688$ m^2, $f_2 = -53.95048$ rad/s, and $f_3 = 0.14499 \times 10^{-8}$. In order to have a common basis for comparison and to avoid working with different objectives in different units, new objective functions F_i are constructed as $F_1 = c_1 f_1$, $F_2 = c_2 f_2$, and $F_3 = c_3 f_3$ and the constants c_1, c_2, and c_3 are selected so that $F_1 = -F_2 = F_3 = 53.95048$ at the starting design vector X_0. The design and constraint vectors corresponding to the initial design are shown in the second column of Table 2.

The implementation of most of the multiobjective optimization techniques requires goals or targets to be set for the individual objective functions. It is convenient to choose the goal of the ith objective as F_i^*, the constrained minimum of F_i obtained without consideration of the other objective functions. The minima F_i^* ($i = 1,2,3$) and the corresponding design and constraint vectors are indicated in Table 2.

The results obtained by using the various multiobjective optimization methods are described below. In all the cases, the basic optimization routine used (for the solution of the single-objective deterministic optimization problem) is the interior penalty function method that incorporates the Davidon-Fletcher-Powell variable metric method of unconstrained minimization and the cubic interpolation method of one-dimensional search.[18]

Global Criterion Formulation

In this formulation the optimum solution X^* is found bv minimizing a preselected global criterion $\underset{\sim}{F}(X)$ such as the sum of the squares of the relative deviations of the individual objective functions from the feasible ideal solutions. Thus X^* is found by minimizing

$$\underset{\sim}{F}(X) = \sum_{i=1}^{k} \left\{ \frac{F_i(X_i^*) - F_i(X)}{F_i(X_i^*)} \right\}^p \tag{30}$$

subject to

$$g_j(X) \leq 0; \quad j = 1,2,...,m_1$$

where p is generally taken as 2 and X_i^* is the feasible ideal solution of the ith objective and is obtained by minimizing $F_i(X)$ subject to the constraints $g_j(X) \leq 0$, $j = 1,2,...,m_1$. Starting with X_0 given in Table 2, which corresponds to a value of $\underset{\sim}{F}(X_0) = 11{,}411.4$ with $p = 2$, the optimum solution was found to be $\underset{\sim}{F}(X^*) = 8.054231$. The results of optimization are shown in Table 3.

Utility Function Formulation

In this formulation, a utility function $U_i(F_i)$ is defined for each objective F_i, depending on the importance of F_i compared to the other objective functions. Then a total or overall utility function U is defined, for example, as

$$U = \sum_{i=1}^{k} U_i(F_i) \tag{31}$$

The solution vector X^* is then found by maximizing the total utility U subject to $g_j(X) \leq 0$, $j = 1,2,...,m_1$. A simple form of Eq. (31) suitable for a maximization formulation is given by

$$U = -\sum_{i=1}^{k} w_i F_i(X) \tag{32}$$

where w_i is a scalar weighting factor associated with the ith objective function. By using the weighting factors $w_i = 1$ ($i = 1,2,3$) and the starting point X_0 with $U(X_0) = -53.950481$, the maximum value of U was found to be $U(X^*) = -15.631068$. The optimum results obtained are indicated in Table 3.

Game Theory Approach

In the game theory approach, the multiobjective optimization problem is viewed as a game problem involving several players, one corresponding to each of the objectives. The system is assumed to be under the control of these intelligent adversaries, each seeking to optimize his/her own gain (objective) at the expense of the opponents using all available information. If the system contains uncertain parameters and random processes, the players do not have perfect information about the state of the system and the problem is called a stochastic game. The basic approach used in applying game theory for minimizing $f_i(X)$ ($i = 1,2,...,k$), subject to the constraints $g_j(X) \leq 0$, $j = 1,2,...,m_1$, and $\ell_j(X) = 0$, $j = 1,2,...,\ell_1$, can be summarized by the following steps[19]:

1) Using X_0 as a starting point, minimize k single criterion optimization problems defined by

$$\text{Minimize } F_i(X)$$

subject to

$$g_j(X) \leq 0, \quad j = 1,2,...,m_1$$

$$\ell_j(X) = 0, \quad j = 1,2,...,\ell_1 \tag{33}$$

and obtain the optimum solutions X_i^*, $i = 1,2,...,k$.

2) With the help of X_i^*, evaluate the elements of the matrix $[P]$ as

$$[P] = \begin{bmatrix} F_1(X_1^*) & F_2(X_1^*) & \cdots & F_k(X_1^*) \\ F_1(X_2^*) & F_2(X_2^*) & \cdots & F_k(X_2^*) \\ \vdots & & & \\ F_1(X_k^*) & F_2(X_k^*) & \cdots & F_k(X_k^*) \end{bmatrix} \tag{34}$$

3) Define the largest element of ith column, F_{iu}, as

$$F_{iu} = \max_{j=1,2,...,k} \{F_i(X_j^*)\}; \quad i = 1,2,...,k \tag{35}$$

4) Construct a supercriterion or bargaining model S as

$$S = \prod_{i=1}^{k} \{F_{iu} - F_i(X_c^*)\} \tag{36}$$

where X_c^* represents the solution (Pareto-optimal solution) of the following problem:

Table 2 Results obtained when individual objective functions are optimized

Quantity	At starting point X_0	At optimum point X_1^* (when F_1 is minimized)	At optimum point X_2^* (when F_2 is minimized)	At optimum point X_3^* (when F_3 is minimized)
Design variables				
x_1	0.25400	0.18442	0.30565	0.35405
x_2	0.45720	0.44830	0.42174	0.31777
x_3	0.00762	0.00618	0.01645	0.03728
Objective functions				
F_1	53.9505	38.9576	119.0881	249.3345
F_2	−53.9505	−43.3350	−73.7592	−73.7214
F_3	53.9505	276.7158	2.7233	0.5004
Mean value of f_1	0.0108394	0.0078271	0.0239263	0.0500946
Constraints				
g_1	−1.0000	-0.4898×10^{-3}[a]	−1.0000	−1.0000
g_2	−0.8798	−0.9812	-0.6090×10^{-3}[a]	-0.1293×10^{-4}[a]
g_3	−0.5520	−0.3461	−0.9182	−0.9960
g_4	−0.1703	−0.3360	−0.7421	−0.9520
g_5	−0.4788	−0.6216	−0.3728	−0.1522
g_6	−0.0618	−0.0801	−0.1346	−0.4026
g_7	−0.1876	−0.1874	−0.1877	−0.1877
g_8	−0.2540	−0.1844	−0.3056	−0.3540
g_9	−0.4572	−0.4483	−0.4217	−0.3178
g_{10}	−0.0076	−0.0062	−0.0164	−0.0373
Supercriterion S	4.62036×10^5	0	10.85734×10^5	0
One-dimensional steps required	—	54	36	37

[a] Active constraint.

$$\text{Minimize } F_c(c,X)=\sum_{i=1}^{k} c_iF_i(X)$$

subject to the constraints

$$g_j(X)\le 0;\quad j=1,2,\ldots,m_1$$
$$\ell_j(X)=0;\quad j=1,2,\ldots,\ell_1$$
$$c_i\ge 0;\quad i=1,2,\ldots,k \tag{37}$$

and

$$\sum_{i=1}^{k} c_i=1$$

where $c=\{c_1, c_2,\ldots,c_k\}^T$. The equality constraint

$$\sum_{i=1}^{k} c_i=1$$

in Eq. (37) can be eliminated by restating the problem of Eq. (37) as follows:

$$\text{Minimize } F_c(c,X)=\sum_{i=1}^{k-1} c_iF_i(X)+\left(1-\sum_{i=1}^{k-1} c_i\right)F_k(X)$$

subject to

$$g_j(X)\le 0;\ j=1,2,\ldots,m_1,\ \ \ell_j(X)=0;\ \ j=1,2,\ldots,\ell_1$$
$$c_i\ge 0;\quad i=1,2,\ldots,k-1$$

and

$$\sum_{i=1}^{k-1} c_i\le 1 \tag{38}$$

where $c=\{c_1, c_2,\ldots, c_{k-1}\}^T$.

5) Maximize the supercriterion S defined by Eq. (36) and find the optimal convex combination of the objective functions, that is, c^* and the corresponding optimum solution of the problem $X^*=X_c^*$. In the present example, the matrix $[P]$ can be identified from Table 1 as

$$[P]=\begin{bmatrix} 38.9576 & -43.3350 & 276.7158 \\ 119.0881 & -73.7592 & 2.7233 \\ 249.3345 & -73.7214 & 0.5004 \end{bmatrix} \tag{39}$$

Thus, the supercriterion S to be maximized becomes

$$S=[249.3345-F_1(X_c)]\,[-43.3350-F_2(X_c)]$$
$$\times[276.7158-F_3(X_c)] \tag{40}$$

where X_c denotes the Pareto-optimal solution of the following problem:

$$\text{Minimize } F_c(X)=c_1F_1(X)+c_2F_2(X)+c_3F_3(X)$$

subject to

$$c_1+c_2+c_3=1$$
$$c_i\ge 0,\quad i=1,2,3 \tag{41}$$

The game theory procedure gave the final values of c_1, c_2, and c_3 as 0.260488, 0.422972, and 0.316540, respectively. The results are shown in Table 3.

Goal Programming Method

In the simplest version of goal programming, the designer sets goals for each objective that he/she wishes to attain. The optimum solution X^* is then defined as the one that minimizes the deviations from the set goals. Thus, the goal programming formulation of the multiobjective optimization problem leads to

$$\text{Minimize } \left\{\sum_{j=1}^{k}(d_j^+ + d_j^-)^p\right\}^{1/p},\quad p\ge 1$$

Table 3 Results of global criterion, utility function, and game theory methods

Quantity	Global criterion method	Utility function method	Game theory method
Design vector X^*	0.47568	0.33694	0.35734
	0.36283	0.48694	0.48678
	0.01663	0.00921	0.00991
Objective functions			
$F_1(X^*)$	138.8015	75.5275	83.2813
$F_2(X^*)$	−73.7077	−69.8593	−73.7633
$F_3(X^*)$	1.1103	9.9629	6.4358
Mean value of f_i	0.027887	0.0151745	0.0167323
Constraints			
g_1	−1.0000	−1.0000	−1.0000
g_2	-0.4603×10^{-2} [a]	−0.2781	-0.1321×10^{-3} [a]
g_3	−0.9407	−0.6520	−0.6997
g_4	−0.3889	-0.2924×10^{-3} [a]	-0.2918×10^{-1} [a]
g_5	-0.2389×10^{-1} [a]	−0.3086	−0.2667
g_6	−0.2555	-0.7863×10^{-3} [a]	-0.1155×10^{-2} [a]
g_7	−0.1877	−0.1877	−0.1877
g_8	−0.4757	−0.3369	−0.3573
g_9	−0.3628	−0.4869	−0.4868
g_{10}	−0.0166	−0.0092	−0.0099
Supercriterion S	9.25258×10^5	12.29760×10^5	13.656449×10^5
One-dimensional steps required	55	53	526

[a] Active constraint.

subject to

$$g_j(X)\le 0,\quad j=1,2,\ldots,m_1$$
$$F_j(X)-d_j^+ + d_j^- = b_j,\quad j=1,2,\ldots,k$$
$$d_j^+\ge 0,\quad j=1,2,\ldots,k$$
$$d_j^-\ge 0,\quad j=1,2,\ldots,k$$
$$d_j^+ d_j^- = 0,\quad j=1,2,\ldots,k \tag{42}$$

where b_j is the goal set by the designer for the jth objective and d_j^+ and d_j^- are, respectively, the under- and overachievement of the jth goal. The value of p is based upon the utility function chosen by the designer.

In the present case, the goals b_i are taken to be the same as F_i^* obtained with individual minimization of F_i (Table 2). Since it is not possible to obtain overachievement of the goals, d_j^- need not be defined for $j=1,2,3$. Thus, the mixed equality-inequality constrained problem stated in Eq. (42) can be restated as an equivalent inequality constrained problem as follows:

$$\text{Minimize } \underset{\sim}{F}(X) = \left[\sum_{j=1}^{3} (d_j^+)^p\right]^{1/p};\quad p\ge 1$$

subject to

$$g_j(X)\le 0,\quad j=1,2,\ldots,10$$
$$d_j^+\ge 0,\quad j=1,2,3 \tag{43}$$

where

$$d_j^+ = F_j(X)-b_j,\quad j=1,2,3$$
$$b_j = F_j^*,\quad j=1,2,3$$

The solution of the problem stated in Eq. (43), with two different choices of p, is given in Table 4. At the starting point X_0, the underachievements are $d_1^+=14.9929$, $d_2^+=19.8087$, and $d_3^+=53.4301$. $F(X_0)=58.9414$ with $p=2$ and 88.2517 with $p=1$. At the optimum solution, $F(X^*_{p=2})=33.9969$ with $p=2$ and $F(X^*_{p=1})=49.8156$ with $p=1$. It is to be noted that this method will be identical to the utility function method when $p=1$ in Eq. (43) and $w_i=-1$ $(i=1,2,3)$ in Eq. (32).

Goal Attainment Method

This method requires the setting up of goals $b_1, b_2,\ldots,b_k$ and weights $w_1, w_2,\ldots,w_k$ for the objective functions $F_1, F_2,\ldots,F_k$, respectively. The weights w_i relate the relative under- or overattainment of the desired goals. Then the following problem is solved to find the solution X^*:

$$\text{Minimize } z$$

subject to

$$g_j(X)\le 0;\quad j=1,2,\ldots,m_1$$
$$F_i(X)-w_i\cdot z\le b_i;\quad i=1,2,\ldots,k$$
$$w_i\ge 0;\quad i=1,2,\ldots,k \tag{44}$$

where z is a scalar variable unrestricted in sign and $w_1, w_2,\ldots,w_k$ are normalized so that

$$\sum_{i=1}^{k} |w_i| = 1 \tag{45}$$

In the case of underattainment of the desired goals, a smaller weighting coefficient is associated with a more important objective. For overattainment of the desired goals, a smaller weighting coefficient is associated with a less important objective.

In the numerical example, all of the weights w_j $(j=1,2,3)$ are assumed to be the same $(=1/3)$ and the goals b_j are taken as F_j^* $(j=1,2,3)$, see Table 2. The initial value of z is assumed as $z_0=300.0$ and the final (optimum) value of z is found to be $z^*=170.4237$. The results are indicated in Table 4. The values of $[F_i(X)-w_i z-b_i]$ are found to be −2.1820, −1.0872, −1.8620 at X_0 and −0.001744, −0.7229, −2.0831 at X^* for $i=1,2,3$.

Table 4 Results obtained with goal programming and goal attaiment methods

Quantity	Goal programming method With $p=2$	With $p=1$	Goal attainment method
Design vector X^*	0.30680	0.33934	0.31527
	0.48286	0.48709	0.44080
	0.00839	0.00927	0.01272
Objective functions			
$F_1(X^*)$	65.9673	76.3062	95.6976
$F_2(X^*)$	−64.1370	−70.2921	−70.2698
$F_3(X^*)$	18.7672	9.5003	5.2317
Mean value of f_1	0.0132537	0.0153309	0.0192269
Constraints			
g_1	−1.0000	−1.0000	−1.0000
g_2	−0.5822	−0.2504	−0.2517
g_3	−0.5879	−0.6567	−0.8504
g_4	-0.1841×10^{-2}[a]	-0.3794×10^{-3}[a]	−0.5409
g_5	−0.3704	−0.3037	−0.3531
g_6	-0.9153×10^{-2}[a]	-0.4746×10^{-3}[a]	-0.9546×10^{-1}[a]
g_7	−0.1876	−0.1877	−0.1877
g_8	−0.3068	−0.3393	−0.3153
g_9	−0.4829	−0.4871	−0.4408
g_{10}	−0.0084	−0.0093	−0.0127
Other quantities d_1^+	27.0097	37.3486	—
d_2^+	9.6221	3.4670	—
d_3^+	18.2668	9.0000	—
Supercriterion S	9.83922×10^5	12.46386×10^5	11.2345×10^5
One-dimensional steps required	41	54	47

[a] Active constraint.

Table 5 Results obtained with bounded objective function method

Quantity	With F_1 as objective	With F_2 as objective	With F_3 as objective
Design vector X^*	0.26139	0.27585	0.28165
	0.46378	0.47643	0.47663
	0.00717	0.00795	0.00795
Objective functions			
$F_1(X^*)$	51.7661	59.5537	59.9976
$F_2(X^*)$	−53.8371	−59.4124	−59.8329
$F_3(X^*)$	59.7767	31.9373	30.2600
Mean value of f_1	0.0104005	0.0119651	0.0120543
Constraints			
g_1	−1.0000	−1.0000	−1.0000
g_2	−0.8818	−0.7532	−0.7407
g_3	−0.4794	−0.5533	−0.5525
g_4	-0.7711×10^{-2}[a]	−0.1015	-0.6233×10^{-1}[a]
g_5	−0.4636	−0.4339	−0.4221
g_6	-0.4832×10^{-1}[a]	-0.2236×10^{-1}[a]	-0.2195×10^{-1}[a]
g_7	−0.1876	−0.1876	−0.1876
g_8	−0.2614	−0.2758	−0.2816
g_9	−0.4638	−0.4764	−0.4766
g_{10}	−0.0072	−0.0079	−0.0079
Constraints of Eq. (46c)	-0.7127×10^{-1}[a]	−0.8321	−0.8333
	−0.8575	-0.7495×10^{-2}[a]	-0.3946×10^{-4}[a]
	−0.8327	−0.6869	−0.1643
	-0.3736×10^{-2}[a]	−0.8787	−0.6713
Supercriterion S	4.50123×10^5	7.46864×10^5	7.69844×10^5
One-dimensional steps required	38	24	55

[a] Active constraint.

Bounded Objective Function Formulation

Here the minimum and the maximum acceptable achievement levels for each objective function F_i are specified as $L^{(i)}$ and $U^{(i)}$, respectively, for $i=1,2,...,k$. Then the optimum solution X^* is found by solving the following problem:

$$\text{Minimize } F_r(X) \tag{46a}$$

subject to

$$g_j(X)\leq 0, \quad j=1,2,...,m_1 \tag{46b}$$

and

$$L^{(i)}\leq F_i(X)\leq U^{(i)}, \quad i=1,2,...,k; \quad i\neq r \tag{46c}$$

The illustrative example is solved by considering F_1, F_2, and F_3 in turn as objective functions in Eq. (46). The lower bounds assumed in the problem are $L^{(1)}=10.0$, $L^{(2)}=100.0$, and $L^{(3)}=10.0$, while the upper bounds are taken as $U^{(1)}=60.0$, $U^2=-50.0$, and $U^{(3)}=60.0$. The optimization results obtained in the three cases are shown in Table 5.

Table 6 Results obtained with lexicographic method

Quantity	At end of stage 1	At end of stage 2	At end of stage 3
Design vector X^*	0.18442	0.18749	0.18822
	0.44830	0.48690	0.48691
	0.00618	0.00595	0.00609
Objective functions			
$F_1(X^*)$	38.9576	39.9305	40.9268
$F_2(X^*)$	−43.3350	−47.3718	−47.9576
$F_3(X^*)$	276.7158	222.7256	201.9022
Mean value of f_1	0.0078271	0.0080225	0.0082228
Constraints			
g_1	-0.4898×10^{-3}[a]	−0.8371	−0.9210
g_2	−0.9812	−0.9587	−0.9542
g_3	−0.3461	−0.1660	−0.2043
g_4	−0.3360	−0.2580	−0.2865
g_5	−0.6216	−0.6153	−0.6138
g_6	-0.8008×10^{-1}[a]	-0.8817×10^{-3}[a]	-0.8543×10^{-3}[a]
g_7	−0.1874	−0.1874	−0.1874
g_8	−0.1844	−0.1875	−0.1882
g_9	−0.4483	−0.4869	−0.4869
g_{10}	−0.0062	−0.0059	−0.0061
$0.975-(F_1/F_1^*)$	−0.0250	−0.0500	−0.0500
$(F_1/F_1^*)-1.025$	−0.0250	-0.2789×10^{-4}[a]	-0.4935×10^{-4}[a]
$0.975-(F_2/F_2^*)$	—	—	−0.0374
$(F_2/F_2^*)-1.025$	—	—	−0.0126
Supercriterion S	0	0.45639×10^5	0.72074×10^5
One-dimensional steps required	54	40	27

[a] Active constraint.

Lexicographic Method

In this method, the objectives are ranked in order of importance by the designer. The optimum solution X^* is then obtained by minimizing the objective functions, starting with the most important one and proceeding according to the order of importance of the objectives.

Let the subscripts of the objectives indicate not only the objective function number, but also the priority of the objective. Thus, $F_1(X)$ and $F_k(X)$ denote the most and least important objective functions, respectively. Then the first problem is formulated as

$$\text{Minimize } F_1(X)$$

subject to

$$g_j(X)\leq 0;\quad j=1,2,\ldots,m_1 \tag{47}$$

and its solution X_1^* and $F_1^*=(X_1^*)$ is obtained. Then, the second problem is formulated as

$$\text{Minimize } F_2(X)$$

subject to

$$g_j(X)\leq 0;\quad j=1,2,\ldots,m_1$$

$$F_1(X)=F_1^* \tag{48}$$

and the solution of this problem is obtained as X_2^* and $F_2^*=F_2(X_2^*)$. This procedure is repeated until all k objectives have been considered. The ith problem is given by

$$\text{Minimize } F_i(X)$$

subject to

$$g_j(X)\leq 0;\quad j=1,2,\ldots,m_1$$

$$F_\ell(X)=F_\ell^*;\quad \ell=1,2,\ldots,i-1 \tag{49}$$

The solution obtained at the end, i.e., X_k^* is taken as the desired solution X^* of the problem.

The example problem is solved in three stages as follows:

1) Starting with X_0, minimize $F_1(X)$, subject to $g_j(X)\leq 0$, $j=1,2,\ldots,10$. The resulting solution is denoted as X_1^* and F_1^*.

2) Starting from X_1^*, minimize $F_2(X)$, subject to $g_j(X)\leq 0$, $j=1,2,\ldots,10$, and $0.975\ F_1^*\leq F_1(X)\leq 1.025\ F_1^*$. The solution of this problem is denoted as X_2^*, $F_1^{**}=F_1(X_2^*)$, and $F_2^{**}=F_2(X_2^*)$.

3) Starting from X_2^*, minimize $F_3(X)$, subject to $g_j(X)\leq 0$, $j=1,2,\ldots,10$, $0.975\ F_1^{**}\leq F_1(X)\leq 1.025\ F_1^{**}$, and $0.975\ F_2^{**}\leq F_2(X)\leq 1.025\ F_2^{**}$.

The results of optimization are given in Table 6.

Discussion

The linearization procedure used in this work for converting probabilistic constraints into equivalent deterministic form can be used for all similar applications. The dynamic probabilistic constraints involving a time parameter have been simplified by assuming the stochastic process to be stationary. This makes the first and second moments of the random response quantities independent of time. This assumption, along with the use of appropriate upper bounds for the stochastic response quantities, appears to be simple and practical for most complex dynamic systems.

The starting design vector X_0 was taken to be the same for all of the multiobjective optimization methods so that their relative efficiencies could be compared. It is to be noted that other attributes, such as simplicity of computer programming, data preparation time required, general applicability of the method, and the usefulness of the optimum solution found, are also to be considered in evaluating the performance of the various methods. Although numerous quantities can be defined for comparing the worth of a multiobjective optimum solution, the supercriterion S defined in the game theory approach is used in this work. Further, the computational effort required in implementing a particular procedure is directly related to the number of one-dimensional minimization steps required in finding the final solution.† As such the values of S and the number of one-dimensional steps are also given at the end of the tabulated results.

†The maximum number of cubic interpolations permitted in each one-dimensional search is limited to four in all the methods.

It can be observed that the game theory approach requires the maximum computational effort. At the same time, the optimum solution found can be seen to be the most valuable in terms of S. Thus, game theory can be considered to be very efficient in obtaining a proper balance between the optimum values of the various objective functions.

The goal programming method using $p=1$ in Eq. (42) gave the next best value of S with only about 10.3% computational effort. The utility function method, which is commonly used as a heuristic approach, also gave a large value of S with only about 10.1% computational effort compared to the game theory. Thus, it appears that goal programming and utility function methods are very efficient from the point of view of computational effort.

The optimum solutions given by the various multicriteria optimization methods can be observed to be different from each other. Since the solutions are different, the set of active constraints will also be different in each case. This is an inherent characteristic of any multicriteria optimization problem; hence, a solution concept or procedure has to be defined on the basis of other attributes such as the mathematical basis of the method, its generality, and the quality of the final solution.

The stress constraint has been observed to be active at the optimum point in the case of the minimization of F_1, while the acceleration requirement was found to be critical in the case of the minimization of F_2 or F_3. The solution found from the game theory has been observed to be critical with respect to the acceleration, web buckling, and upper bound on $\bar{d}$. In the case of the global criterion method, only the constraints on the acceleration and the upper bound on $\bar{d}$ were found to be active at the optimum point. The solutions given by the utility function and goal programming were critical with respect to the web buckling and upper bound on $\bar{d}$. The constraint on the first objective function was active in the case of the goal attainment method. In the bounded objective function technique, the web buckling and upper bound on $\bar{d}$ have been found to be critical at the optimum point when F_1 or F_3 were considered directly for minimization, while the upper bound on $\bar{d}$ was found to be the only active constraint when F_2 was considered for minimization. In addition, some of the bounds set on the objective functions, other than the one considered for minimization, were also found to be active.

When the lexicographic method was used for multiobjective optimization, the upper bounds on the stress and $\bar{d}$ were found to be critical at the end of stage 1; but only the latter remained active at the end of stages 2 and 3. If the number of active behavior constraints at the optimum solution were taken as a criterion, the game theory method can be considered to be superior to all other methods.

The present investigation suggests that the game theory approach is preferred for the solution of multiobjective problems as it provides not only the optimum design vector, but also the optimal combination of the various objective functions. However, the computational effort is maximum in this approach. Hence, future efforts have to be directed toward reducing the computational effort. Also, the potentialities of goal programming and utility function methods need to be investigated with reference to other multicriteria structural optimization problems.

Appendix: Standard Deviations of Response Quantities

The standard deviations of the response quantities are obtained by using the partial derivative rule, in conjunction with Taylor's series expansion,[20] as

$$\sigma_{f_1}^2 = 2\bar{h}^2(\sigma_b^2+\sigma_d^2)+2(\bar{b}+\bar{d})\sigma_h^2 \tag{A1}$$

$$\sigma_\omega^2 = [(3\bar{E}\bar{I})^2(\bar{\ell}^6\sigma_m^2+9\bar{\ell}^4\bar{m}^2\sigma_\ell^2) + \bar{m}^2\bar{\ell}^6(9\bar{E}^2\sigma_I^2+9\bar{I}^2\sigma_E^2)]^{1/2}/(\bar{m}^2\bar{\ell}^6) \tag{A2}$$

$$\sigma_I = [9(\bar{d}^2\bar{h}-2\bar{d}\bar{h}^2)^2\sigma_b^2+9(2\bar{b}\bar{d}\bar{h}-2\bar{b}\bar{h}^2 + \bar{d}^2\bar{h}-4\bar{d}\bar{h}^2)^2\sigma_d^2+(3\bar{b}\bar{d}^2-12\bar{b}\bar{d}\bar{h} + \bar{d}^3-12\bar{d}^2\bar{h})^2\sigma_h^2]^{1/2}/6 \tag{A3}$$

$$\bar{I} = (3\bar{b}\bar{d}^2\bar{h}-6\bar{b}\bar{d}\bar{h}^2+\bar{d}^3\bar{h}-6\bar{d}^2\bar{h}^2)/6 \tag{A4}$$

$$\sigma_S = \frac{3\bar{E}\bar{d}}{2\bar{\ell}^2}\cdot\sigma_Y \tag{A5}$$

$$\sigma_Y = \left(\frac{\pi\Phi_0}{2\zeta\omega^3}\right)^{1/2} \tag{A6}$$

$$\sigma_{S_f} = k_1(4\bar{E}^2\bar{h}^4\bar{b}^2\sigma_b^2+\bar{b}^4\bar{h}^4\sigma_E^2+4\bar{E}^2\bar{b}^4\bar{h}^2\sigma_h^2)^{1/2} \tag{A7}$$

$$\sigma_{S_w} = k_2(4\bar{E}^2\bar{h}^4\bar{d}^2\sigma_d^2+\bar{d}^4\bar{h}^4\sigma_E^2+4\bar{E}^2\bar{d}^4\bar{h}^2\sigma_h^2)^{1/2} \tag{A8}$$

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